Gauging Lie superalgebras

1980 J. Phys. A: Math. Gen. 13 1537

(http://iopscience.iop.org/0305-4470/13/5/013)

J. Phys. A: Math. Gen. **13** (1980) 1537–1542. Printed in Great Britain

Gauging Lie superalgebras

C D Pickup and J G Taylor

Department of Mathematics, King's College, University of London, Strand, London WC2R 2LS, UK

Received 2 July 1979

Abstract. We present arguments for the detailed form of Lagrangian to be selected in gauging Z_2-graded Lie superalgebras, with particular reference to $\mathrm{SU}(n|m)$. On physical grounds the use of the graded trace is unacceptable. The ordinary trace is not invariant under the whole superalgebra, though we find that a condition of 'first-order' gauge invariance can be satisfied. Certain restrictions on the gauge potentials then result, which we enumerate most specifically for $\mathrm{SU}(2|1)$.

A Lie superalgebra (Z_2-graded Lie algebra) (Kac 1977, Scheunert *et al* 1977 and references therein) is a generalisation of a Lie algebra in which some of the generators obey anticommutation relations:

$$[Q_a, Q_b]_- = t^c_{ab}Q_c \qquad [Q_a, R_\alpha]_- = t^\beta_{a\alpha}R_\beta \qquad [R_\alpha, R_\beta]_+ = t^c_{\alpha\beta}Q_c.$$

In a more compact notation let $\{T_a\} = \{Q_a, R_\alpha\}$, and define the degree $\deg(T)$ of a generator by $\deg(Q_a) = 0$, $\deg(R_\alpha) = 1$. Then the generalised commutation relations can be written

$$[T_a, T_b] = T_aT_b - (-1)^{\deg(T_a)\deg(T_b)}T_bT_a \tag{1}$$

or more simply $[T_a, T_b] = T_aT_b - (-1)^{ab}T_bT_a$ (Kac 1977), where $a = \deg(T_a)$. From this definition it is easy to show that

$$[T_a, [T_b, T_c]](-1)^{ac} + [T_b, [T_c, T_a]](-1)^{ba} + [T_c, [T_a, T_b]](-1)^{cb} = 0 \tag{2}$$

which is the generalised Jacobi identity.

Relations (1) and (2) define a Lie superalgebra, A. A is the direct sum of two parts, $A = A_0 + A_1$, where A_0 contains the even elements, Q_a, and A_1 contains the odd elements R_α. We can decompose a general $a \in A$ as $a = a_0 + a_1$ where $a_i \in A_i$. An endomorphism, D, of the algebra is called a derivation of degree d if

$$D(ab) = D(a)b + (-1)^{ad}aD(b) \qquad \text{where} \qquad a, b, \in A.$$

For example, in a gauge transformation on a (non-graded) Lie algebra, D might be a gauge transformation $D(a) = [D, a]_-$ where $D, a \in A_{(0)}$. In a matrix representation the even elements consist of matrices of block diagonal form

$$\left(\begin{array}{c|c} a & 0 \\ \hline 0 & d \end{array}\right)$$

where a, d are square matrices, whilst the odd elements are

$$\left(\begin{array}{c|c} & b \\ \hline c & \end{array}\right).$$

We define the graded trace by

$$\mathrm{tr_g}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{tr}\ a - \mathrm{tr}\ d. \tag{3}$$

We now consider the problem of gauging a Lie superalgebra. In the mathematical literature it is customary to regard gauge transformations as derivations and for invariant forms to be defined in terms of the graded trace. An approach has been suggested (Taylor 1979a) taking the gauge transformation in the non-graded sense. However, this requires stronger restrictions on the gauge potentials, along the lines discussed below, than that resulting from the use of graded gauge transformations. Since the latter gives covariantly transforming field strengths, and hence a simpler structure, we will pursue it here. We note however that there is a difference in the structure of the Higgs fields in the ungraded as compared to the graded case: in six-dimensional space–time there is only the single traditional massive Higgs particle in the former case as compared to five (two charged, three neutral) such in the latter case.

Returning to the graded case we define

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{4}$$

where A_μ takes values in the Lie superalgebra,

$$A_\mu = A_\mu^a T_a.$$

Under a gauge transformation

$$\delta_u A_\mu = \partial_\mu u - [u, A_\mu] \tag{5}$$

$$\begin{aligned}\delta_u F_{\mu\nu} &= -\partial_\mu[u, A_\nu] + \partial_\nu[u, A_\mu] + [\partial_\mu u - [u, A_\mu], A_\nu] + (-1)^{uA_\mu}[A_\mu, \partial_\nu u - [u, A_\nu]] \\ &= -[u, F_{\mu\nu}]\end{aligned}$$

where we have used the Jacobi identity (2). The covariant derivative is defined analogously to that in Lie algebras,

$$D_\mu = \partial_\mu + A_\mu. \tag{6}$$

Let ψ be a set of fields transforming as

$$\delta_u \psi = -u\psi.$$

Then

$$\begin{aligned}\delta_u(D_\mu\psi) &= (\partial_\mu u - [u, A_\mu])\psi + (\partial_\mu + (-1)^{uA_\mu} A_\mu)(-u\psi) \\ &= -u(D_\mu\psi)\end{aligned}$$

i.e. $D_\mu\psi$ transforms covariantly.

Next we consider invariant forms. A well-known invariant bilinear form is $\mathrm{tr_g}\,(AB)$ (Kac 1977). (We do not distinguish between an element of the superalgebra and its representation.) This is invariant under $\delta_u A = [u, A]$ since

$$\begin{aligned}\delta_u \operatorname{tr}_g(AB) &= \operatorname{tr}_g([u, A]B + (-1)^{uA}A[u, B]) \\ &= \operatorname{tr}_g(uAB - (-1)^{u(A+B)}ABu) \\ &= 0 \qquad \text{since} \qquad \operatorname{tr}_g(XY) = (-1)^{XY} \operatorname{tr}_g(YX).\end{aligned}$$

Thus $\operatorname{tr}_g(F_{\mu\nu}F^{\mu\nu})$ is an invariant quadratic form. If

$$F_{\mu\nu} \sim \begin{pmatrix} a & b \\ d & c \end{pmatrix}$$

then $\operatorname{tr}_g(F^2) \sim \operatorname{tr}(a^2 - c^2)$.

Specifically, when we look at the generalisations of $SU(2) \times U(1)$ (Dondi and Jarvis 1978, Fairlie 1979, Ne'eman 1979, Taylor 1979a), both a and c are either Hermitian or antiHermitian, and so $\operatorname{tr}(a^2 - c^2)$ is physically unacceptable due to negative energies arising in $\operatorname{tr}(c^2)$. $\operatorname{tr}_g(FF^+)$ would also have this problem.

We thus reject $\operatorname{tr}_g(F_{\mu\nu}F^{\mu\nu})$ as a physically acceptable kinetic energy term and turn to the ordinary trace. In this we shall be much more specific and consider the Lie superalgebra $SU(m|n)$ whose matrix representations are of the form

$$A = \begin{pmatrix} a & b \\ ib^+ & c \end{pmatrix}$$

according to Freund and Kaplansky (1976); a, c are antiHermitian, so $A_0^+ = -A_0$ and $A_1^+ = -iA_1$ (Berezin (1977) is similar except $A_1^+ = iA_1$). Fairlie (1979) and Taylor (1979a) have a different definition:

$$A = \begin{pmatrix} a & ib \\ ib^+ & c \end{pmatrix} \qquad a^+ = a,\ c^+ = c. \tag{7}$$

Thus $A_0^+ = A_0$, $A_1^+ = -A_1$, so that the bracket has to be modified. If $A, B \in SU(m|n)$ then

$$i[A, B] \in SU(m|n)$$

where

$$[A, B] = [A_0, B_0]_- + [A_0, B_1]_- + [A_1, B_0]_- + i[A_1, B_1]_+.$$

Consider first the Freund and Kaplansky representation:

$$\operatorname{tr}(F^2) \sim \operatorname{tr}(a^2 + c^2 + 2ib^+b).$$

a^2, c^2 and b^+b are real and so this Lagrangian is not real and thus it is unacceptable. We need not examine its invariance. To overcome this problem we could try $\operatorname{tr}(FF^+)$ which is real but positive definite. Thus if we hope to include the Higgs fields in this model and have spontaneous symmetry breaking we cannot use $\operatorname{tr}(FF^+)$ as a Lagrangian. This objection also rules out $\operatorname{tr}(FF^+)$ for representation (7) as well. (This case has been examined in Taylor (1979b).)

Finally we come to $\operatorname{tr}(F_{\mu\nu}F^{\mu\nu})$ in representation (7). This is real, and not positive definite, and thus might contain some interesting physics.

First we must examine its invariance. To avoid confusion with signs we proceed as follows:

$$\delta_u \operatorname{tr}(AB) = \mathrm{i} \operatorname{tr}([u, A)B + (-1)^{uA} A[u, B])$$
$$= \mathrm{i} \operatorname{tr}\{(u_1 A_0 B_1 - u_1 B_1 A_0 + u_1 A_1 B_0 - u_1 B_0 A_1)$$
$$+ \mathrm{i}(u_1 A_1 B_0 + u_1 B_0 A_1 + u_1 B_1 A_0 + u_1 A_0 B_1)\}.$$

Now $A = B = F$, giving

$$\delta_u \operatorname{tr}(FF) = \mathrm{i} \operatorname{tr}\{(u_1 F_0 F_1 - u_1 F_1 F_0 + u_1 F_1 F_0 - u_1 F_0 F_1)$$
$$+ \mathrm{i}(u_1 F_1 F_0 + u_1 F_0 F_1 + u_1 F_1 F_0 + u_1 F_0 F_1)\}$$
$$= -2 \operatorname{tr}\{u_1 (F_0 F_1 + F_1 F_0)\}. \tag{8}$$

If $[F_0, F_1]_+ = 0$ then tr(FF) is invariant. We extend the dimensionality of space–time from 4 to n. and denote a general index by capital Roman letters. $g_{MN} = \operatorname{diag}(+1, -1, -1, -1, -1, \ldots -1)$. Let μ, ν denote indices from 1 to 4 and m, n indices from 5 to n. (8) reads in full:

$$[F_{0MN}, F_1^{MN}]_+ = 0 \tag{9}$$

where any fields are independent of the additional dimensions and the summation convention is being used. To regain conventional gauge theory in four dimensions we choose

$$F_{1\mu\nu} = 0 \qquad F_{0\mu\nu} \neq 0. \tag{9a}$$

Split $[F_{0MN}, F_1^{MN}]_+$ into parts:

$$[F_{0MN}, F_1^{MN}]_+ = [F_{0\mu\nu}, F_1^{\mu\nu}]_+ + [F_{0\mu n}, F_1^{\mu n}]_+ + [F_{0m\nu}, F_1^{m\nu}]_+ + [F_{0mn}, F_1^{mn}]_+$$

and look now at the two middle terms:

$$F_{\mu n} = \partial_\mu A_n + \mathrm{i}[A_\mu, A_n]_-$$
$$= -F_{n\mu}.$$

So we need only examine one of these terms:

$$[F_{0\mu n}, F_1^{\mu n}]_+ = (\partial_\mu A_{0n} + \mathrm{i}[A_\mu, A_{0n}]_-)(\partial^\mu A_1^n + \mathrm{i}[A^\mu, A_1^n]_-)$$
$$+ (\partial_\mu A_{1n} + \mathrm{i}[A_\mu, A_{1n}]_-)(\partial^\mu A_0^n + \mathrm{i}[A^\mu, A_0^n]_-).$$

This will satisfy (8) if (a) A_{0m} is a constant which commutes with A_μ, or (b) A_{1m} is constant and commutes with A_μ. Examination of the generators reveals that (b) can only be satisfied if $A_{1m} = 0$. Condition (a) is satisfied by

$$A_{0m} = M_m \lambda_{8'} \tag{9b}$$

where M_m is a constant. We are left with the term

$$[F_{0mn}, F_1^{mn}]_+. \tag{10}$$

$$F_{0mn} = \mathrm{i}([A_{0m}, A_{0n}]_- + \mathrm{i}[A_{1m}, A_{1n}]_+) = F_{0nm}$$

since $\lambda_{8'}$ commutes with itself.

$$F_{1mn} = \mathrm{i}([A_{0m}, A_{1n}]_- + [A_{1m}, A_{0n}]_-) = -F_{1nm}.$$

(10) can be written as

$$\sum_{m\leqslant n}([F_{0mn}, F_1^{mn}]_+ + [F_{0nm}, F_1^{nm}]_+) = \sum_{m\leqslant n}([F_{0mn}, F_1^{mn}]_+ - [F_{0mn}, F_1^{mn}]_+)$$
$$= 0.$$

Thus equations ($9a$) and ($9b$) ensure the invariance of the Lagrangian $\mathrm{tr}(F_{MN}F^{MN})$ under infinitesimal transformations of the superalgebra.

We now turn to SU(2|1) in six space–time dimensions and choose

$$A_5 = \begin{pmatrix} m_5 & \mathrm{i}\phi \\ \mathrm{i}\phi^+ & 2m_5 \end{pmatrix} \qquad A_6 = \begin{pmatrix} m_6 & \mathrm{i}\psi \\ \mathrm{i}\psi^+ & 2m_6 \end{pmatrix}$$

where ϕ, ψ are 2×1 column vectors. Thus:

$$F_{\mu 5} = \begin{pmatrix} 0 & \mathrm{i}D_\mu\phi \\ \mathrm{i}(D_\mu\phi)^+ & 0 \end{pmatrix} \qquad F_{\mu 6} = \begin{pmatrix} 0 & \mathrm{i}D_\mu\psi \\ \mathrm{i}(D_\mu\psi)^+ & 0 \end{pmatrix}$$

$$F_{55} = \begin{pmatrix} 2\phi\phi^+ & 0 \\ 0 & 2\phi^+\phi \end{pmatrix} \qquad F_{66} = \begin{pmatrix} 2\psi\psi^+ & 0 \\ 0 & 2\psi^+\psi \end{pmatrix}$$

$$F_{56} = \begin{pmatrix} \phi\psi^+ + \psi\phi^+ & m_5\psi - m_6\phi \\ -(m_5\psi^+ - m_6\phi^+) & \phi^+\psi + \psi^+\phi \end{pmatrix}.$$

We can now examine the contribution of the Higgs fields to the Lagrangian:

$$\mathscr{L}_{\mathrm{H}} = -\tfrac{1}{4}\mathrm{tr}(2F_{\mu 5}F^{\mu 5} + 2F_{\mu 6}F^{\mu 6} + F_{55}F^{55} + F_{56}F^{56} + F_{65}F^{65} + F_{66}F^{66})$$

where the extra dimensions are space-like as before:

$$\mathscr{L}_{\mathrm{H}} = -|D_\mu\phi|^2 - |D_\mu\psi|^2 - 2(\phi^+\phi)^2 - 2(\psi^+\psi)^2 - \tfrac{1}{2}((\psi^+\phi)^2 + (\phi^+\psi)^2 + 2|\psi|^2|\phi|^2)$$
$$-\tfrac{1}{2}(\phi^+\psi + \psi^+\phi)^2 + |m_5\psi - m_6\phi|^2.$$

The sign of the kinetic energy term is incorrect. However, if we choose a metric which is time-like in the extra dimensions (Taylor 1979), then

$$\mathscr{L}_{\mathrm{H}} = |D_\mu\phi|^2 + |D_\mu\psi|^2 - 2|\phi^+\phi|^2 - 2(\psi^+\psi)^2 - \tfrac{1}{2}((\psi^+\phi)^2 + (\phi^+\psi)^2 + 2|\psi|^2|\phi|^2)$$
$$-\tfrac{1}{2}(\phi^+\psi + \psi^+\phi)^2 + |m_5\psi - m_6\phi|^2.$$

Analysis of the potential for the case $m_5 = m_6$ shows that for there to be a minimum of the potential

$$\langle\psi\rangle_0 = -\langle\phi\rangle_0$$

which reproduces Fairlie's condition (Fairlie 1979), though now only on $\langle A_5\rangle_0$ and $\langle A_6\rangle_0$. Thus we achieve spontaneous symmetry breaking. Matter can be coupled to this system, as discussed in Fairlie (1979), Dondi and Jarvis (1979) and Taylor (1979b, 1980).

We conclude by remarking that an invariant gauging of the complete superalgebra has not been achieved, since (9) is not itself gauge-invariant; for example, the solutions ($9a$) will acquire odd components in a different gauge. Having fixed the odd part u_1 of the gauge transformation u then there is indeed invariance under even gauge transformations. For SU($n|m$) we thus have the remaining *true* gauge invariance under

$SU(n)\times SU(m)$. We can thus regard the invariance condition on the Lagrangian $\mathscr{L}$:

$$\delta_u \mathscr{L} = 0 \tag{10}$$

as a means of specifying the odd element of $\mathscr{L}$, and thereby the potentials A_M through condition (9). We can regard (10) as a condition of *first-order gauge invariance*: the Lagrangian is to be chosen to be invariant under infinitesimal gauge transformations δ_u for u in the full superalgebra. The resulting classical theory is then to be quantised in the resulting non-trivial potentials.

This situation is not completely satisfactory, but at worst we can regard it as a prescription for restricting the Higgs and fermion structure of the truly gauge-invariant $SU(n)\times SU(m)$ theory remaining after imposing (10). In particular, various restrictions are (Taylor 1980): (i) the Higgs multiplets can only occur as fundamental representations of $SU(n)$, and at most one for each extra time-like dimension, (ii) all non-zero Higgs masses are equal (to 150 BeV in $SU(2|1)$), (iii) all lepton masses must be below 54 BeV, (iv) each extra heavy lepton requires two extra time-like dimensions. Such results may well be tested in the next few years. In the meantime it would appear of value to attempt to gauge the superalgebra properly; we hope to return to that elsewhere.

Acknowledgments

C D Pickup would like to thank the Science Research Council of Great Britain for financial support and J G Taylor would like to thank A Salam for various helpful discussions.

References

Berezin F A 1977 *Lie Superalgebras* Moscow ITEP-66
Dondi P H and Jarvis P D 1978 *Preprint* University of Southampton THEP 78/9-8
Fairlie D B 1979 *Phys. Lett.* B **82** 97
Freund P G O and Kaplansky I 1976 *J. Math. Phys.* **17** 228
Kac V G 1977 *Commun. Math. Phys.* **53** 31
Ne'eman Y 1979 *Phys. Lett.* B **81** 190
Scheunert M, Nahm W and Rittenberg V 1977 *J. Math. Phys.* **18** 146
Taylor J G 1979a *Phys. Lett.* **83** B 331
—— 1979b *Phys. Lett.* **84** B 79
—— 1980 *J. Phys. A: Math. Gen.* **13**